数理解析研究所講究録 1222 巻 2001 年 86-89

Finitely Generated Idempotent-free Semilattice-Indecomposable Semigroups with Relations I

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A semigroup S is called $\mathcal{S}-$indecomposable if S has no semilattice homomorphic image except the trivial semilattice. We assume $S \neq S^2$, $|S\backslash S^2| < \infty$ and S is generated by $S\backslash S^2$. Let $B = S\backslash S^2 = \{a_1 \ldots, a_k\}$. The purpose of this paper is to report the structure of idempotent-free $\mathcal{S}-$indecomposable semigroup S generated by B with relation as defined below. Let Z_+ be the set of all positive integers. We assume

(1) $$a_1^{m_1} = \cdots = a_k^{m_k} \quad \text{for some } m_1, \ldots, m_k \in Z_+.$$

In particular we study here the free semigroup satisfying (1), that is, every such semigroup is a homomorphic image of the free one. The condition (1) is so strong that the property of $\mathcal{S}-$indecomposability is derived from (1).

Lemma 1. *If S is a semigroup generated by B and satisfies (1), then S is $\mathcal{S}-$indecomposable.*

Since $|B| < \infty$ the condition (1) is equivalent to (1$'$) below.

(1$'$) For each pair $a_i, a_j \in B$ there exist $n_i, n_j \in Z_+$ such that

$$a_i^{n_i} = a_j^{n_j}.$$

If B satisfies (1$'$), equivalently (1), we say S is *power jointly generated* by B.

Let S be an idempotent-free semigroup which is power-jointly generated by B with (1), and let F be the free semigroup over B. There is a homomorphism $f : F \to S$ which satisfies the following conditions

i) $X \in F,\ a \in B,\ f(X) = f\{(a)\} \Rightarrow X = \{a\}$.

ii) $f(a_1)^{m_1} = \cdots = f(a_k)^{m_k}$.

Let ρ denote the congruence on F generated by the set of binary relations

$$\{(a_i^{m_i}, a_j^{m_j}) : a_i, a_j \in B\}.$$

Then S is a homomorphic image of F/ρ keeping every element of B fixed. In this paper we study F/ρ. For simplicity of notation, let $S = F/\rho$, so $X\rho Y$ in F if and only if $X = Y$ in S.

From (1) we immediately have

Lemma 2. $a_i^{\lambda m_i} a_j^x = a_j^x a_i^{\lambda m_i}$ *for* $i, j = 1, \ldots, k$, *for any* $\lambda \in Z_+$.

Let $X \in S$. X has the form $X = a_{i_1}^{x_1'} \cdots a_{i_s}^{x_s'}$ where $a_{i_j} \in B$, $(j = 1, \ldots, s)$ $x_i' \in Z_+$, (2) $a_{i_j} \neq a_{i_{j+1}}$, $(j = 1, \ldots, s-1)$.

We rewrite $x_i' = x_i + \lambda_i m_i$ where $0 < x_i \leq m_i$, $\lambda_i \in Z_+^0 = Z_+ \cup \{0\}$. Let $M = a_1^{m_1} = \cdots = a_k^{m_k}$. By using Lemma 2 repeatedly we have

(3) $$X = a_{i_1}^{x_1} \cdots a_{i_s}^{x_s} a_{i_1}^{\lambda_1 m_{i_1}} \cdots a_{i_s}^{\lambda_s m_{i_s}} = a_{i_1}^{x_1} \cdots a_{i_s}^{x_s} M^\lambda \text{ where } \lambda = \lambda_1 + \cdots + \lambda_s.$$

Likewise $Y = a_{j_1}^{y_1} \cdots a_{j_t}^{y_t} a_{j_1}^{\mu_1 m_{j_1}} \cdots a_{j_t}^{\mu_t m_{j_t}} = a_{j_1}^{y_1} \cdots a_{j_t}^{y_t} M^\mu$ where $\mu = \mu_1 + \cdots + \mu_t$.

Consider the product XY. Again by using Lemma 2 we have:

If $i_s \neq j_1$ $\quad XY = a_{i_1}^{x_1} \cdots a_{i_s}^{x_s} a_{j_1}^{y_1} \cdots a_{j_t}^{y_t} M^{\lambda+\mu}$.

If $i_s = j_1$ and $x_s + y_1 \leq 2m_{i_s}$, then $XY = a_{i_1}^{x_1} \cdots a_{i_{s-1}}^{x_{s-1}} a_{i_s}^{z_s} a_{j_2}^{y_2} \cdots a_{j_t}^{y_t} M^{\lambda+\mu}$ where $0 < z_s \leq m_{i_s}$ and $z_s \equiv x_s + y_1 \pmod{m_{i_s}}$.

If $i_s = j_1$ and $x_s + y_1 > 2m_{i_s}$, then $XY = a_{i_1}^{x_1} \cdots a_{i_{s-1}}^{x_{s-1}} a_{i_s}^{z_s} a_{j_2}^{y_2} \cdots a_{j_t}^{y_t} M^{\lambda+\mu+1}$ where $0 < z_s \leq m_{1_s}$ and $z_s \equiv x_s + y_1 \pmod{m_{i_s}}$.

Let P denote the set of finite sequences V of elements of B, $V = a_{i_1} \cdots a_{i_s}$ satisfying $a_{i_j} \neq a_{i_{j-1}}$, $\quad j = 1, \ldots, s-1$.

The binary operation on P is defined by

$$(a_{i_1} \cdots a_{i_s}) * (a_{j_1} \cdots a_{j_t}) = \begin{cases} a_{i_1} \cdots a_{i_s} a_{j_1} \cdots a_{j_t} & \text{if } i_s \neq j_1 \\ \\ a_{i_1} \cdots a_{i_s} a_{j_2} \cdots a_{j_t} & \text{if } i_s = j_1 \end{cases}$$

that is, if $i_s \neq j_1$, the product is juxtaposition, if $i_s = j_1$ then one of a_{i_s} and a_{j_1} is omitted.

Proposition 1. *P is a semigroup and S is homomorphic onto P under the mapping* $a_{i_1}^{x_1} \cdots a_{i_s}^{x_s} \to a_{i_1} \cdots a_{i_s}$.

P is regarded as the set of finite sequences $i_1 \cdots i_s$ of elements of $B = \{1, \ldots, k\}$ subject to $i_j \neq i_{j+1}$, $j = 1, \ldots, s-1$, $s \geq 1$. In the form (3): $X = a_{i_1}^{x_1} \cdots a_{i_s}^{x_s} M^\lambda$, we rewrite x_j by x_{i_j} $(j = 1, \cdots, s)$

(3$'$) $X = a_{i_1}^{x_{i_1}} \cdots a_{i_s}^{x_{i_s}} M^{\lambda}$.

The sequence $x_{i_1} \cdots x_{i_s}$ is regarded as a mapping from a sequence $i_1 \cdots i_s$ of elements of $\{1, \ldots, k\}$ to a sequence $x_{i_1} \cdots x_{i_s}$ such that $x_{i_j} \in Z_{m_{i_j}}$ (i.e. an element modulo m_{i_j}) and $0 < x_{i_j} \leq m_{i_j}$, $j = 1, \ldots, s$, $s = 1, \ldots, k$. Let $\varphi : i_1 \cdots i_s \to x_{i_1} \cdots x_{i_s}$, $\psi : j_i \cdots j_s \to y_{j_1} \cdots y_{j_s}$ and let Φ denote the set of all such φ's and define the binary operation $\varphi\psi$ on Φ as follows:

If $i_s \neq j_1$, $(i_1 \cdots i_s) * (j_1 \cdots j_t) = i_1 \cdots i_s j_1 \cdots j_t \to x_{i_1} \cdots x_{i_s} y_1 \cdots y_{j_t}$.

If $i_s = j_1$, $(i_1 \cdots i_s) * (j_1 \cdots j_t) = i_1 \cdots i_{s-1} i_s j_2 \cdots j_t \to x_{i_1} \cdots x_{i_{s-1}} z_{i_s} y_{j_2} \cdots y_{j_t}$, where $z_{i_s} \equiv x_{i_s} + y_{j_1} (\bmod m_{i_s})$, $0 < z_{i_s} \leq m_{i_s}$.

Proposition 2. *Φ is a semigroup, and S is homomorphic onto Φ under the mapping $X = a_{i_1}^{x_{i_1}} \cdots a_{i_s}^{x_{i_s}} M^{\lambda} \to \varphi$ where $\varphi : i_1 \cdots i_s \to x_{i_1} \cdots x_{i_s}$.*

Define a mapping $g : \Phi \times \Phi \to Z_+^0$ as follows:

$$g(\varphi, \psi) = \begin{cases} 1 & \text{if } i_s = j_1 \text{ and } x_{i_s} + y_{j_1} > m_{i_s} \\ 0 & \text{otherwise} \end{cases} .$$

Let $\Gamma = \{(\varphi, \lambda) : \varphi \in \Phi,\ \lambda \in Z_+^0\}$ and define the binary operation on Γ as follows:

$$(\varphi, \lambda)(\psi, \mu) = (\varphi\psi,\ \lambda + \mu + g(\varphi, \psi)).$$

Note that g satisfies the condition:

$$g(\varphi, \psi) + g(\varphi\psi, \xi) = g(\varphi, \psi\xi) + g(\psi, \xi) \quad \text{for all } \varphi, \psi, \xi \in \Phi.$$

Now we have the main theorem

Theorem . *Γ is a semigroup and S is isomorphic onto Γ under the mapping $X = a_{i_1}^{x_{i_1}} \cdots a_{i_s}^{x_{i_s}} M^{\lambda} \to (\varphi, \lambda)$ where $\varphi : i_1 \cdots i_s \to x_{i_1} \cdots x_{i_s}$.*

The idea of constructing $S-$indecomposable semigroups from a certain free semigroup was initiated by the author in case of finite nil semigroups 1958 [2], and also the idea was used in case of finitely generated $Z-$semigroups [3].

The representation of S by means of Γ is similar to $N-$semigroups (i.e. idempotent-free cancellative commutative archimedean semigroups) [1].

Acknowledgement . *The author expresses heartfelt appreciation to Professor Kunitaka Shoji for advice and presentation on behalf of the author at the Research Conference.*

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